The Towers of Hanoi in Algebra

by Louis Lo Bosco

During the three weeks I spent in the Leadership Program, I was awakened to many interesting topics in Discrete Math. My school already offers a one-semester elective in Discrete Mathematics, taught by Lou D'Angelo (LP '91), so my goal has been to implement the topics throughout the curriculum.

In my Honors Algebra II course, a sophomore-level course (with a few freshman), I decided to introduce the concept of recursion. I remembered how impressed I was with Lead Teacher Phil Reynolds' (LP '89) presentation on the Towers of Hanoi, and decided to try it. First I needed some working models. I found some old 1-by-2-foot boards lying around, cut the boards into 8-inch lengths and hammered three nails with small heads into each board. I used six metal washers of decreasing size for a set of discs.

I put the students into groups of four and told them the classic story: "According to an ancient Hindu legend, Brahma piled 64 gold disks one on top of the other. The disks, each a different size, were stacked in order with the largest at the bottom and the smallest at the top. Priests were told to transfer the disks one at a time from one pile to another, using a third pile if necessary, so that at no stage would a larger disk be placed on top of a smaller disk. When the work was complete, according to the legend, the world would end."

I gave students a worksheet (see sidebar), gave them time to investigate, find the results, then record and describe their solutions.

The students really seemed to enjoy this hands-on work. All the groups eventually figured out a recursion formula for the number of moves,and everyone realized that the time needed to move all 64 disks was extremely large*. The students enjoyed the problem enough that I decided to venture one step further. All of my students use the TI-81 calculator, so I asked them to write a program to compute the number of moves needed using their recursive formula. Most of them had never written a program before, but as a group, we established a correct algorithm; I then helped them to enter the program into the calculator. The students were impressed with how simple and short the actual program was.

This project took about three days to complete, and was well worth it. I was most impressed with those groups that said that even though we could write down a mathematical procedure to move all 64 disks, we could not actually perform this task in the real world. They had understood the difference between a theoretical construction and a practical one.

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* 585 billion years approximately!

Towers of Hanoi Student Worksheet

Objective: move all disks from Tower 1 to Tower 3, using Tower 2 as needed. Rules:
a. Move only one disk at a time.
b. Never put a larger disk on top of a smaller one.
c. Try to use the minimum number of moves.

1. Record your moves using first 3 disks, then 4, then 5, using a table like the one below, which is an example for 2 disks. A is the larger disk: AB in the column labeled T1 means that B is on top of A on the first tower.

Move #	T1	T2	T3
0	AB	-	-
1	A	B	-
2	-	B	A
3	-	-	AB

2. Look for a pattern, and try to find a general solution that will work for any number of disks.

3. Explain how you arrived at this solution.

4. Assume that you can move one disk a second. Determine the number of years it would take to move the 64 disks (the number of years it would take for the world to end according to the legend).

What conclusions can your group draw from this exercise? In particular, what did this exercise help you to understand? What conclusions can you draw from part 4?